MATH 320 Unit 0 Exercises

Divisibility and Primes in $\mathbb{Z}$

Given $m, n \in \mathbb{Z}$, we say that m *divides* n, writing $m|n$, if there is some $k \in \mathbb{Z}$ with $mk = n$. We then call m a *divisor* of n.

Given $r, s, t \in \mathbb{Z}$, we say that r is a *common divisor* of s and t if r is a divisor of both, i.e. if $r|s$ and $r|t$.

Given $p \in \mathbb{Z}$ with $p \notin \{-1, 0, 1\}$, we say that p is *prime* if it satisfies:

$$\forall a, b \in \mathbb{Z},\ \text{if } p|ab \text{ then } \big(p|a \text{ or } p|b\big).$$

For Aug. 28:

1. Let $a, b \in \mathbb{Z}$ with $b \neq 0$ and $a|b$. Prove that $|a| \leq |b|$.
2. Let $a, b, c \in \mathbb{Z}$ with $a|b$, $b|c$, and $c|a$. Prove that $|a| = |c|$.
3. Let $a, b, c, q, r \in \mathbb{Z}$ with $a = bq + r$. Prove that c is a common divisor of a, b if and only if c is a common divisor of b, r.
4. Let $a, b, p \in \mathbb{Z}$ with p prime and $p = ab$. Prove that $|a| = 1$ or $|b| = 1$.

Extra:

5. Let $a, b, c \in \mathbb{Z}$. Prove that c is a common divisor of a, b, if and only if c is a common divisor of $|a|, |b|$.
6. Let $n \in \mathbb{Z}$ with $n \geq 1$, and let $a, b_1, b_2, \ldots, b_n \in \mathbb{Z}$ with $a|b_1$, $a|b_2, \ldots, a|b_n$. Prove that $a|(b_1 + b_2 + \cdots + b_n)$.
7. Let $n \in \mathbb{Z}$ with $n \geq 1$, and let $a_1, a_2, \ldots, a_n, p \in \mathbb{Z}$ with p prime and $p|a_1 a_2 \cdots a_n$. Prove that there is at least one $i \in \{1, 2, \ldots, n\}$ with $p|a_i$.

Comments: Problems 1-4 will have solutions presented by four students on Thursday (sign up using the Google sheet linked in Canvas). On Tuesday Sep. 2 (one week from today) will be our Unit 0 exam. There will be four questions, at least three of which will be quite similar to these seven exercises. No notes, calculators, or other aids are permitted. Ample paper will be provided, as well as the contents of the double box above.

You may need to use one exercise to help you solve another (later) one. You may also do this on the exam, but only if that exercise is added to the double box as an available theorem.

Hint for problems 6,7: induction.